Technical Notes

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Modal Equations for the Nonlinear Flexural Vibrations of Plates

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I. Introduction

THE coupled nonlinear equations of motion based on the von Karman theory of plates can be derived using Hamilton's principle.[1] A slightly modified form of these equations is the so called dynamic analog of the von Karman equations.[1-4] An alternative formulation of the von Karman equations consists of an equilibrium equation for the transverse motion and a compatibility equation for the inplane stresses. The aim of the present Note is to systematically consider higher-order Galerkin approximations to the solutions of the inplane equations of motion as well as the inplane compatibility equation for an assumed single mode expression for w. The computational technique outlined in Ref. 3 is adopted here. The present treatment confines to the assumed space mode approach. The modal equations based on these approximations as well as the approximate solution of the transverse equation of motion are discussed with the help of two typical plate geometries. The significant role of the stretching due to inplane displacements, which has sometimes been overlooked in the literature, is highlighted with the help of the present results. Some of the earlier investigations are considered in the light of the present investigation.

II. Analysis

The three nonlinear equations of motion of a plate expressed in terms of the two inplane displacements u and v, and the transverse displacement w can be found from Ref. 1 or 4. This formulation will henceforth be referred to as the (u,v,w) formulation. An alternative form of the von Karman equations, expressed in terms of the transverse displacement w and Airy's stress function F, is well known and may be found from Ref. 4 [note that the sign associated with E_2 in Eq. (2) of Ref. 4 should read as positive]. This formulation will henceforth be referred to as the (w,F) formulation. In the analysis to follow, for a given set of boundary conditions, the type of formulation has been chosen in such a way that the construction of coordinate functions becomes simpler.

Considering the (w,F) formulation, for an assumed single mode expansion for w, a higher-order Galerkin approximation to the solution of the inplane compatibility equation can be written as

$$F=\Sigma\theta_i f_i \text{ and } [K_F]\{\theta\}=\{Q\}\tau^2 \tag{1}$$

where τ and θ_i are the generalized coordinates in the expansions for w and F, respectively. f_i are the coordinate functions, $[K_F]$ is a known coefficient matrix obtained from the weighted residuals of the compatibility equation, and $\{Q\}$ originates from the nonlinear terms in the compatibility equation. Applying the Galerkin technique to the transverse equilibrium equation and using Eq. (1), one gets the following single modal equation governing the nonlinear flexural vibration of plates

$$\ddot{\tau}+A_1\tau+A_2\tau^3+A_3=0 \tag{2}$$

where A_1 and A_3 are constants for an assumed w; A_3 is associated with the transverse loading term. The nonlinear coefficient A_2 depends on Eq. (1) as well as the coordinate function for w. It may be noted that the above method can still be used with a multimode expansion for w which, however, will result in a complicated system of modal equations instead of Eq. (2). The method outlined here has already been used[3] in conjunction with the (u,v,w) formulation with multimode expansions for u and v. For this case, the approximate solutions will be obtained both for the von Karman equations and the direct variational equations of motion as explained in Ref. 3. An n term approximation would imply n term expansions for each of the variables u, v, or F.

From the structure of the equations in the (u,v,w) formulation,[3,4] one can easily identify the contributions to the coefficient A_2 under two categories: a) the nonlinear terms involving the derivatives of w alone; and b) the nonlinear terms involving the derivatives of u and w, and v and w. It is obvious that the second category is associated with the stretching due to the inplane displacements caused by nonlinear coupling between the inplane and transverse motions. In the case of the (w,F) formulation, if the stretching due to the transverse slopes is known, A_2 can be decomposed as above.

The nonlinear flexural vibrations of two sets of plate problems, namely, rectangular plates and isosceles triangular plates, are considered in the present investigation. The modal equation is based on the n term approximation to the solution of either the inplane equilibrium equations or the compatibility equation for an assumed single mode expansion for w. The analytical details of these examples are summarized next. The coordinate functions satisfy the relevant boundary conditions.

A. Clamped Rectangular Plates (Fig. 1a)

$$w=\tau f_1^2 \tag{3}$$

where $f_1=(X^2-1)\ (Y^2-1)$, and $X=x/a$, $Y=y/b$. The various inplane boundary conditions along with the type of formulation used are listed as follows:

Case 1: Constrained, Type I (u,v,w)

$$u=v=0 \text{ at the boundaries} \tag{4}$$

$$u=Xf_1\langle\theta_1,\ \theta_2,\ \ldots\theta_8\rangle\ \langle P_1\rangle^T$$

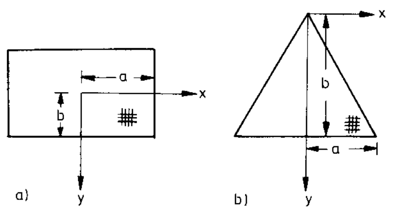

Fig. 1 Geometry of the plate.

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$$v = Y f_1 \langle \lambda_1, \lambda_2, \ldots \lambda_8 \rangle \langle P_1 \rangle^T \tag{5}$$

where f_1 is defined in Eq. (3), and

$$\langle P_1 \rangle = \langle 1, X^2, Y^2, X^2Y^2, X^4, Y^4, X^2Y^4, X^4Y^2 \rangle$$

τ, θ_i, and λ_i are the generalized coordinates. This case has been considered in Ref. 3 also.

Case 2: Constrained, Type II (u,v,w)

$$u = \partial^2 F/\partial x\, \partial y = 0 \text{ at } X = \pm 1,\; v = \partial^2 F/\partial x \partial y = 0 \text{ at } Y = \pm 1 \tag{6}$$

$$u = X(X^2 - 1)\, g_1(X,Y) \tag{7}$$

where

$$g_1(X,Y) = \theta_1 + \theta_2\, g_2(Y) + \theta_3\, X^2 + \theta_4 X^4$$
$$+ g_2(Y)\,(\theta_5 X^2 + \theta_6 Y^2 + \theta_7 X^2 Y^2 + \theta_8 X^4 + \theta_9 Y^4)$$

and $g_2(Y) = (Y^2 - 1)^2$. The corresponding expansion for v is obtained by interchanging X and Y, and replacing θ_i with λ_i.

Case 3: Unconstrained (w,F)

$$\partial^2 F/\partial y^2 = \partial^2 F/\partial x \partial y = 0 \quad \text{at } X = \pm 1$$

$$\partial^2 F/\partial x^2 = \partial^2 F/\partial x \partial y = 0 \quad \text{at } Y = \pm 1 \tag{8}$$

$$F = f_1^2 (\theta_1 + \theta_2 X^2 + \theta_3 Y^2 + \theta_4 X^2 Y^2 + \theta_5 X^4 + \theta_6 Y^4) \tag{9}$$

where f_1 is defined in Eq. (3).

B. Freely Supported Rectangular Plates (Fig. 1a)

$$w = \tau (X^4 - 6X^2 + 5)\;(Y^4 - 6Y^2 + 5)/25 \tag{10}$$

The three sets of inplane boundary conditions are the same as those given by Eqs. (4, 6, and 8). The respective series for u, v, and F are given by Eqs. (5, 7, and 9) with the relevant f_1 being identified from Eq. (3).

C. Clamped Isosceles Triangular Plates (Fig. 1b)

$$w = \tau (729\, f_1^2 / 16) \tag{11}$$

where $f_1 = (X^2 - Y^2)\;(Y - 1)$, and X and Y are defined as in Eq. (3) with the relevant a and b.

Case 1: Constrained, Type I, (u,v,w)

$$u = v = 0 \text{ at the boundaries} \tag{12}$$

$$u = X f_1 \langle \theta_1, \theta_2, \ldots \theta_{12} \rangle \langle P_2 \rangle^T$$

$$v = f_1 \langle \lambda_1, \lambda_2, \ldots \lambda_{12} \rangle \langle P_2 \rangle^T \tag{13}$$

where

$$\langle P_2 \rangle = \langle 1, Y, X^2, Y^2, X^2Y, Y^3, X^2Y^2, Y^4, X^2Y^3, X^4Y, X^4Y^2, X^4Y^3 \rangle$$

This case has also been considered in Ref. 3 with a slightly different set of coordinates functions.

Case 2: Unconstrained (w,F)

$$\partial^2 F/\partial s^2 = \partial^2 F/\partial s \partial n = 0 \text{ at the edges} \tag{14}$$

$$F = f_1^2 (\theta_1 + \theta_2 Y + \theta_3 X^2 + \theta_4 Y^2 + \theta_5 X^2 Y + \theta_6 Y^3) \tag{15}$$

where f_1 is defined in Eq. (11). A one term counterpart of Eq. (15) has been considered in Ref. 4.

III. Numerical Results

The polynomial integrals involved in the application of the Galerkin technique to the problems discussed in Sec. II. have been exactly evaluated with the help of a computer code called NONPL1. Considering the linear parts of the various solutions obtained (see Table 1), the result for the clamped rectangular plate coincides with the one discussed by Bert.[5] For an isotropic square plate with free supports, the square of the natural frequency given by the present polynomial solution is proportional to 2.23196, the exact value being proportional to 2.23006. The linear frequency for the clamped isosceles triangular plate with $a/b = 1$ is about 17% higher than the value reported in Ref. 6.

In the case of numerical examples solved using the (u,v,w) formulation, the value of A_2 has been obtained as an average of the results based on the von Karman and the variational

Table 1 Coefficients of the modal equations for nonlinear plates

Plate geometry		Constrained, type I Orthotropic	Constrained, type I Isotropic	Constrained, type II Orthotropic	Constrained, type II Isotropic	Unconstrained Orthotropic	Unconstrained Isotropic	($\rho a^4 A_1/h^2 \times 10^5$) kg/cm²
Rectangle,	$\bar{A}_2$[a]	42.605	3.5771	42.305	3.5138	5.3590	1.1582	Orthotropic:
clamped	$\bar{u}\bar{v}$	25.097	1.9508	24.797	1.8875	-12.149	-0.4681	87.029
$a/b=1$	$\bar{w}$	17.508	1.6263	17.508	1.6263	17.508	1.6263	Isotropic:
		(75,0.2,50)[b]	(75,0.2,49)	(75,0.4,51)	(80,0.4,48)	(25,0.8)[c]	(16,0.6)	7.4176
		$n=8$		$n=9$		$n=6$		
Rectangle,	$\bar{A}_2$	35.653	2.9830	35.646	2.9694	1.6310	0.38286	Orthotropic:
freely	$\bar{u}\bar{v}$	19.036	1.6121	19.029	1.5985	-14.986	-0.98804	20.416
supported	$\bar{w}$	16.617	1.3709	16.617	1.3709	16.617	1.3709	Istrotropic:
$a/b=1$		(15,0.1,8)	(17,0.1,10)	(40,0.0,25)	(45,0.00,30)	(4,0.0)	(3,0.0)	2.2320
		$n=4$		$n=6$		$n=6$		
Isosceles	$\bar{A}_2$	476.11	106.37	...	...	89.740	10.339	Orthotropic:
triangle	$\bar{u}\bar{v}$	290.20	53.297	...	...	-96.167	-42.738	1213.2
clamped	$\bar{w}$	185.91	53.077	...	....	185.91	53.077	Isotropic:
$a/b=1$		(71,3.5,51)	(70,6.5,50)	...	...	(40,0.5)	(45,2.5)	275.00
		$n=12$				$n=6$		

[a] $\bar{A}_2 = \rho a^4 A_2 10^5$ kg/cm²; $\bar{u}\bar{v}$ stretching due to inplane displacements; $\bar{w}$ stretching due to transverse slopes. ρ is the mass density.
[b] The first and second values in the bracket are respectively the percentage differences in one term bounds and n term bounds; the third value is the percentage difference between one term and n term Rayleigh-Ritz solutions.
[c] The first value in the the bracket is the percentage difference between one term and n term solutions and the second value is the percentage difference between $(n-1)$ term and n term solutions.

Table 2 Comparison of nonlinear coefficient $\bar{A}_2$ with the previous results for isotropic material

Geometry of the plate	Inplane boundary conditions: Constrained type I	Constrained type II	Unconstrained	Average of B.C. in column 3	Reference[a]
	...	3.9771	1.6130	1.8027	9
Rectangle,	...	6.0959	2.1792	...	8
Clamped	3.5771	3.5138	1.1582	...	Present
Rectangle,	...	2.9353	0.3952	0.7610	2,9 and 7
freely	...	...	0.1903	...	7
supported	...	2.9730	0.4648	...	8
	2.9830	2.9694	0.3829	...	Present

[a] All other results except the present ones are based on trigonometric modes for w.

equations, which were seen to be the two bounds for the exact value of A_2 for a given w. Such a bounding property has already been reported.[3] Typical numerical values of the nonlinear coefficient A_2 for various cases considered in Sec. II are given in Table 1 for $a/b=1$. These values are decomposed into two parts for the case of dynamic analog of the von Karman equations as discussed earlier. The accuracies of the various solutions obtained are also summarized in Table 1. The Galerkin approximations to the solutions of the inplane compatibility equation converged from below as expected. The Young's moduli for the orthotropic material are $E_1=28 \times 10^5$ kg/cm^2 and $E_2=2.24 \times 10^5$ kg/cm^2; the Poisson's ratios are $\nu_1=0.2$ and $\nu_2=0.016$; the shear modulus is $G_{12}=2.24\times 10^5$ kg/cm^2. The properties of the isotropic material are $E_1=1\times 10^5$ kg/cm^2 and $\nu_1=0.3$.

IV. Discussion

The values of A_2 in Table 1 clearly show the significant role played by the stretching due to the inplane displacements (i.e., *uv* stretching). Also, it is seen that in the case of plates with unconstrained boundary conditons, the inplane displacements introduce a predominantly compressive strain field, thus yielding a negative value for the *uv* stretching. It is conjectured that a re-examination of the Berger approximation to the nonlinear plate equations, with the role of *uv* and *w* stretching components being duly identified, might yield some interesting results. An extensive computation for a few aspect ratios revealed that the first-order Galerkin approximations to the solutions of the inplane equilibrium equations may yield a small *uv* stretching contribution in many cases, even though the accurate solutions yielded significant *uv* stretching components. Hence, Bert's[5] assumption of neglecting the *uv* stretching is not justified.

Considering the results in Table 2, it is seen that for the two types of constrained boundary conditions for a rectangular plate, the values of A_2 given by the present analysis do not differ appreciably. Similar observations were made by Dowell and Ventres[7] also. The present results for the constrained clamped rectangular plates are better than the previous results, because the polynomial mode for w is less stiff than the trigonometric mode used by others.

Considering the case of plates with edges free of inplane stress resultants (unconstrained), all other results in Table 2 (except the present results) are, in general, based on less accurate solutions to the inplane compatibility equations. However, the deviations between the present and previous results are both due to the aforementioned error and the different type of mode used for w. It has been observed that the first-order Galerkin approximations can have substantial error in the case of plates with unconstrained boundary conditions also. Although an averaged condition for the case of the unconstrained boundary condition has not been considered in the present analysis, the results in Table 2 clearly show the possible wide difference between these two conditions as observed by Dowell and Ventres;[7] only a one-term Galerkin approximation was obtained however. Finally, a comparison of the results due to Bayeles et al.[8] with various other results clearly shows the inadequacy of a first-order Galerkin approximation to the solution of the inplane equilibrium equations or the compatibility equation.

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Approximate Shock-Free Transonic Solution for Lifting Airfoils

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Introduction

PRACTICAL and mathematical interests in transonic potential flow problems have grown in recent times through works of Nieuwland[4] and his co-workers in NLR, The Netherlands, and the experimental works of Holder and Pearcy[5] in England. Among the various analytical methods based on transonic small perturbation theory for studying transonic profile flow problem, the integral equation method of Oswatitsch, improved and extended further by Gullstrand, Spreiter, and Alksne,[2] is well known.

Nörstrud[6] extended the integral equation method of Oswatitsch to lifting flows by the finite difference method. Recently, Subramanian and Balakrisnan[7] extended the technique of local linearization method to lifting airfoils.

The present Note gives an extension of the simple approximate shock-free transonic solution of the integral equation of Oswatitsch given by Niyogi and Mitra[8] to the lifting case. Numerical results for parabolic arc profiles and the NACA 0012 profile at different angles of attack have been compared with previous analytical, numerical, and experimental results.

Basic Equations and Solution

Consider a steady, inviscid, non-heat-conducting, plane

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